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TECHNICAL NOTE

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THE THEORETICAL ENTHALPY DISTRIBUTION OF AIR IN STEADY FLOW ALONG THE AXIS OF A DIRECT-CURRENT ELECTRIC ARC

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THE THEORETICAL ENTHALPY DISTRIBUTION OF AIR IN STEADY FLOW ALONG THE AXIS OF A DIRECT-CURRENT ELECTRIC ARC

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SUMMARY

An approximate solution is obtained for the enthalpy and electrical conductivity distributions in a cylindrical, direct-current-arc column with steady air flow along the axis. The only form of energy loss considered is lateral heat conduction. The solution predicts the following behavior for given length and diameter of column: (1) higher operating efficiencies at lower pressures and higher mass flows, (2) higher enthalpies at higher currents and lower flow rates, and (3) higher arc voltages with higher mass flows. The solution furthermore indicates that with given mass flow rate, pressure, and current, higher enthalpies occur with increase of length or decrease of radius of the column, that there are simultaneous increases in arc voltages and radial heat transfer, and that the efficiencies will suffer decreases. What has been said with regard to trends with length at fixed mass flow is equivalent to trends with the reciprocal of the mass flow at fixed length. The trend of increasing enthalpy output as radius is decreased agrees with experiment.

INTRODUCTION

No theory is available for predicting the energy transfer within an arc column in a moving stream of air. It is obvious that such a theory would be desirable to provide an understanding of the process by which the air is heated, of how its energy is distributed in space, and of the efficiency of the energy exchange processes. Unfortunately, a treatment of the complete equations that govern the energy transfer (i.e., the equations of motion, Maxwell's equations, and the equation of state) involves such a large number of nonlinearly related variables as to defy ready solutions. Thus, in what follows, the point of view will be that valid conclusions can be drawn from a study of an arc column model simplified sufficiently to permit solution, but retaining the essential and dominant characteristics of the energy transfer processes.

Such a model for an arc column in still air was formulated by Elenbaas and Heller (ref. 1) by neglecting all forms of heat loss except radial heat conduction. Maecker (ref. 2) measured spectroscopically the characteristics of a static arc in nitrogen and concluded that the simplified theoretical model did, indeed, reflect the observed characteristics at moderate arc pressures. The model formulated in this report to represent the direct-current-arc column therefore incorporates the principal assumption that conduction is the dominant means of energy transfer and that radiation loss is unimportant. In addition, the model is assumed to be a right circular cylinder with air in steady flow along the axis. The configuration is meant to represent, for example, the current-carrying region of the wall-stabilized arc-jet plasma generator (shown in fig. 1) in which electrode processes can be neglected. It is further assumed that the boundary layer does not encroach on the current-carrying core.

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The energy equation then uncouples from the equations of motion and Maxwell's equations. Solutions can be obtained for the enthalpy distribution if the further assumptions are incorporated that the air is in local thermodynamic equilibrium and that the Lorentz force is small compared to the applied voltage potential. Such a model permits an approximate solution that is analytic and that predicts the several effects of varying the column radius, column length, current, mass-flow rate, and pressure.

SYMBOLS

A	parameter of the approximation $\sigma = A\varphi$, $\frac{\text{mho-sec}}{\text{Btu}}$
C_c	conversion constant, 9.47×10^{-4}, $\frac{\text{Btu}}{\text{watt-sec}}$
C_n	constants of integration
c_p	specific heat at constant pressure, Btu/lb $^{\circ}$F
D	diameter, ft
E	voltage gradient, volts/ft
$f(y)$	the function $(1-e^{-11.5y/z_o})^{1/2}$, dimensionless
H	average enthalpy, Btu/lb (reference, $H = 0$ at 0° R)
h	enthalpy, $\int c_p\,dT$, Btu/lb (reference, $h = -3{,}500$ at 0° R)
$\tilde{h}$	enthalpy, $\int c_p\,dT$, Btu/lb (reference, $\tilde{h} = 0$ at 0° R)
h_o	the quantity $\frac{c_p I}{k r_e}\left(\frac{C_c}{A}\right)^{1/2}$, Btu/lb

I	current, amp
J_o	zero-order Bessel function of the first kind
J_1	first-order Bessel function of the first kind
j	current density, amp/ft^2
k	thermal conductivity, Btu/ft ^oF sec
l	arc column length, ft
p	pressure, lb/ft^2
Q	total heat transfer from the arc column, Btu/sec
q	local heat transfer from the surface of the arc column, Btu/sec ft^2
r	radial distance from the axis of the arc column, ft
r_e	radius of the current carrying cylinder, ft
T	temperature, ^oR
t	time, sec
u	axial velocity, ft/sec
V	arc column voltage drop, volts
$\dot{w}$	weight flow rate, lb/sec
z	axial distance along the column, ft
z_o	the length $\dot{w}c_p/\pi k$, ft
β	constant of the eigenfunctions, dimensionless
η	efficiency of ohmic heating in the arc column, dimensionless
ρ	density, lb/ft^3
σ	electric conductivity, mho/ft
φ	conductivity function, $\int k\,dT$, Btu/sec ft (reference, $\varphi = -0.3$ at 0^o R)
$\tilde{\varphi}$	conductivity function, $\int k\,dT$, Btu/sec ft (reference, $\tilde{\varphi} = 0$ at 0^o R)
Ω	column resistance, ohm

Subscripts and Superscripts

$(\ )_\infty$ value at large z

$(\ )_E$ radially averaged value at the column exit

$(\ )_r$ partial differentiation with respect to r

$(\ )_z$ partial differentiation with respect to z

$(\ )_w$ value at the solid boundary

$(\bar{\ })$ dimensionless quantity

ANALYSIS

Assumptions

The model used to represent the current-carrying region of the arc column of figure 1 is detailed in figure 2. The shape is a right circular cylinder of length l and radius r_e with coincident axial electric current and air flow. Cylindrical polar coordinates r and z are used within the cylinder. The origin is arbitrarily located at a point along the axis where the electric conductivity is small, so that essentially nonconducting air flows into the region of the discharge to be studied. The boundary condition for the radius is that the electric conductivity also is small, and can be considered precisely zero at the surface $r = r_e$. Thus no electric current flows outside the cylinder of radius r_e. The radial heat flow crossing the boundary must therefore be that function of z which preserves the cylindrical column shape.

The characteristics approximated and the assumptions inherent in this model are as follows:

1. The air flow is assumed one-dimensional, steady, laminar, and axisymmetric with the specific weight flow (ρu) constant.

2. The electric discharge is stationary.

3. Heat loss due to thermal conduction is assumed much larger than radiation heat loss.

4. The air is assumed to be in thermodynamic equilibrium.

5. Lorentz force is negligible.

6. The electric potential is constant on planes perpendicular to the axis.

7. Viscous dissipation and kinetic energy changes are assumed negligible compared to enthalpy changes due to ohmic heating.

The assumptions listed in (1) and (2) above might seem unrealistic since most arc columns in flowing gas appear to be turbulent or, if laminar, unstable. In fact, Kovasznay (ref. 3) has speculated that the flow in all high-current-arc columns must be turbulent, and Cann (ref. 4) describes arc column instabilities of the sausage and kink types which are theoretically inherent in laminar arc columns. However, arc discharges that appear to be stationary and laminar have been produced in commercially available, wall-stabilized arc-jet plasma generators. Furthermore, these generators are claimed to produce very high enthalpies and, therefore, are of particular interest. The theoretical model of figure 2 is expected to be a reasonable approximation to such laminar arc columns.

The remaining assumptions were selected for the following reasons. Finkelnburg (ref. 1) showed that the radiation loss from an arc column is a small fraction of the total loss for column pressures of the order of one atmosphere. Therefore, the dominant form of heat transfer remaining for consideration is conduction, when this conduction includes the heat transfer due to recombination of atoms, ions, and electrons as given by Hansen (ref. 5). The assumption that the air is in thermodynamic equilibrium is valid for densities high enough so that the thermodynamic gradients are small over a mean free path. Cann (ref. 4) shows that this assumption is valid for pressures of one atmosphere or greater. Lorentz force can be shown to be small compared to pressure forces of interest if no external magnetic field is applied. The assumption of constant electric potential on planes perpendicular to the axis simplifies the analysis, but imposes the condition that the direction of current flow is fixed. Lastly, the heating from viscous dissipation and the changes in kinetic energy will be small compared to ohmic heating for high enthalpy arc columns in which the exit flow is choked by an aerodynamic throat. Thus, the pressure will be constant within a factor of two at most. The arc column model shown in figure 2, therefore, is believed to be a reasonable approximation to the arc column of wall-stabilized, laminar arc-jet plasma generators for moderate pressures in the absence of external magnetic fields.

Energy Equation for the Arc Column

For the model assumed here the energy equation can be separated from the equations of motion and Maxwell's equations. Its simultaneous solution with the equation of state and Ohm's law gives the distribution for enthalpy (or electrical conductivity) in the column. After truncation in accordance with the foregoing assumptions the energy equation is:

$$\rho u \frac{\partial \tilde{h}}{\partial z} = \frac{C_c j^2}{\sigma} + \frac{\partial^2 \tilde{\varphi}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\varphi}}{\partial r} + \frac{\partial^2 \tilde{\varphi}}{\partial z^2} \qquad (1)$$

where

$$\tilde{\varphi} = \int k \, dT$$

The first term is the rate of energy absorption by the air, the second term is the rate of energy production by ohmic heating, and the remaining terms give the rate of heat loss by conduction.

Ohm's law for a cylindrical column is

$$j = \sigma E$$

If E is considered independent of radius, it follows that

$$\frac{j}{\sigma} = E(z) = \frac{\int_0^{r_e} 2\pi r j \, dr}{\int_0^{r_e} 2\pi r \sigma \, dr}$$

Therefore, the j^2/σ term of the energy equation can be written as

$$\frac{j^2}{\sigma^2}\,\sigma = \frac{\left(\int_0^{r_e} 2\pi r j \, dr\right)^2}{\left(\int_0^{r_e} 2\pi r \sigma \, dr\right)^2}\,\sigma = \frac{I^2 \sigma}{\left(\int_0^{r_e} 2\pi r \sigma \, dr\right)^2}$$

and the energy equation becomes

$$\rho u \frac{\partial \tilde{h}}{\partial z} = \frac{C_c I^2 \sigma}{\left(\int_0^{r_e} 2\pi r \sigma \, dr\right)^2} + \frac{\partial^2 \tilde{\varphi}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\varphi}}{\partial r} + \frac{\partial^2 \tilde{\varphi}}{\partial z^2} \tag{2}$$

where, from Kirchoff's law of conservation of current, $I = \text{const.}$

The equation of state, $h = h(T,p)$ for air in thermodynamic equilibrium, can be used to define the transport properties σ and $\tilde{\varphi}$ as

$$\sigma = \sigma(T,p), \quad \tilde{\varphi} = \tilde{\varphi}(T,p)$$

since both are considered to be scalars and independent of electromagnetic field strength. For a given constant pressure these can be written as

$$\sigma = \sigma(\tilde{\varphi})$$

and

$$\tilde{h} = \tilde{h}(\tilde{\varphi})$$

where

$$\tilde{\varphi} = \tilde{\varphi}(r,z)$$

These relations have been derived from data in reference 5 by Hansen and reference 6 by Viegas and Peng, and are shown in figures 3 and 4. Since the dependent variables $\tilde{h}$ and σ are both expressible in terms of the conductivity function $\tilde{\varphi}$, equation (2) can be written

$$\rho u \frac{\partial \tilde{h}(\tilde{\varphi})}{\partial z} = \frac{C_c I^2 \sigma(\tilde{\varphi})}{\left(\int_0^{r_e} 2\pi r \sigma(\tilde{\varphi})\ dr\right)^2} + \frac{\partial^2 \tilde{\varphi}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\varphi}}{\partial r} + \frac{\partial^2 \tilde{\varphi}}{\partial z^2} \tag{3}$$

The enthalpy or electric conductivity distributions can be obtained from numerical solutions of this equation. However, if further approximations are made regarding the $\sigma(\tilde{\varphi})$ and $\tilde{h}(\tilde{\varphi})$ curves, an analytic solution can be obtained as is done in the next section. Equation (3) reduces to the Elenbaas-Heller equation if the term on the left (the rate of energy absorption by the air) and the last term on the right (the axial conduction term) are zero.

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Approximate Solution of the Energy Equation

An approximate solution to equation (3) can be obtained if for the nonlinear $\sigma = \sigma(\tilde{\varphi})$ and $\tilde{h} = \tilde{h}(\tilde{\varphi})$ curves a number of piecewise-linear curves are substituted as was done by Goldenberg (ref. 7) for the Elenbaas-Heller energy equation. The errors resulting from applying various numbers of linear segments are discussed in appendix A. As is shown therein, even the rough approximation that the $\tilde{h} = \tilde{h}(\tilde{\varphi})$ and $\sigma = \sigma(\tilde{\varphi})$ curves are single straight lines for appropriate ranges of σ and $\tilde{h}$ is sufficiently accurate to predict over-all trends, except, perhaps, the trends with pressure. The approximation is therefore introduced that enthalpy and electric conductivity are both linear with the conductivity function $\tilde{\varphi}$ and can be represented at a given pressure by single straight lines.

The equations of the lines are $h = (c_p/k)(p)\varphi$ and $\sigma = A(p)\varphi$ where (c_p/k) and A are average slopes determined for given pressure from the curves in figures 3 and 4. For convenience, the origins of the curves in figures 3 and 4 are hereafter shifted so that the integrals $\varphi = \int_{T_1}^{T_2} k\ dT$ and $h = \int_{T_1}^{T_2} c_p\ dT$ are zero when $\sigma = 0$. Thus, the base temperature, T_1, for $h = 0$ and $\varphi = 0$ is the temperature at which σ is taken as zero in the linear approximation. These points are indicated on the curves of figures 3 and 4 at $\tilde{\varphi} \doteq 0.3$ and $\tilde{h} \doteq 3500$ Btu/lb.

Owing to the various approximations and assumptions that have been introduced, the following relations exist between the electric and thermodynamic variables:

$$\varphi = \frac{k}{c_p}(p)h = \frac{\sigma}{A(p)} = \frac{j}{A(p)E(z)}$$

Because E, the electric field strength has been assumed to be a function only of z, and because the total current I is assumed constant, the current density j, at most, can be a function of radius. It therefore follows that at constant pressure the enthalpy h and the conductivity σ must be functions of the type

$$h \sim \sigma \sim R(r)Z(z)$$

The approximate energy equation in terms of enthalpy is

$$[\rho u]\,\frac{\partial h}{\partial z} = \left[C_c\,\frac{c_p}{Ak}\,I^2\right]\frac{h}{\left(\int_0^{r_e} 2\pi r h\;dr\right)^2} + \left[\frac{k}{c_p}\right]\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\,\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial z^2}\right) \qquad (4)$$

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where the terms in square brackets are parameters. The energy equation can also be written in terms of the electric conductivity and the resulting equation will be identical in form to equation (4).

Equation (4) can be made dimensionless by the following change of variables:

$$\bar{z} = z/z_o$$

where

$$z_o = \rho u r_e^2 (c_p/k) \equiv \dot{w} c_p/\pi k$$

$$\bar{r} = r/r_e$$

where r_e is the radius of the current-carrying cylinder, and

$$\bar{h} = h/h_o$$

where

$$h_o = \frac{c_p I}{k r_e}\left(\frac{C_c}{A}\right)^{1/2}$$

Equation (4) reduces to the following dimensionless form:

$$\bar{h}_{\bar{z}} = \frac{\bar{h}}{\left(\int_0^1 2\pi\bar{r}\bar{h}\;d\bar{r}\right)^2} + \left[\frac{r_e}{z_o}\right]^2 \bar{h}_{\bar{z}\bar{z}} + \bar{h}_{\bar{r}\bar{r}} + \frac{1}{\bar{r}}\,\bar{h}_{\bar{r}} \qquad (5)$$

This equation is an elliptic partial differential integral equation which is nonlinear. The term $\left[\frac{r_e}{z_o}\right]^2 h_{\bar{z}\bar{z}}$ represents the loss by axial conduction; so it is seen that the requirement for neglecting axial conduction is that the following inequality hold:

$$\left[\frac{r_e}{z_o}\right]^2 \bar{h}_{\bar{z}\bar{z}} \ll \frac{1}{\bar{r}}\,\bar{h}_{\bar{r}} + \bar{h}_{\bar{r}\bar{r}}$$

The ratio r_e/z_o is easily computed to be much less than unity for arc columns with a radius of less than one foot and a mass flow rate greater than 0.01 lb/sec in the pressure range from 10^{-3} to 10^2 atmospheres. An upper limit for $\bar{h}_{\bar{z}\bar{z}}$ can be obtained a posteriori from the solution for h where axial conduction is neglected. When this is done one finds that only near $z = 0$ is the term $\bar{h}_{\bar{z}\bar{z}}$ the same order of magnitude as $(1/\bar{r})\,\bar{h}_{\bar{r}} + \bar{h}_{\bar{r}\bar{r}}$. Therefore, the term $[r_e/z_o]^2\bar{h}_{\bar{z}\bar{z}}$ is negligible for sufficiently long arc columns and equation (5) becomes

$$\bar{h}_{\bar{z}} = \frac{\bar{h}}{\left(\int_0^1 2\pi\bar{r}\bar{h}\,d\bar{r}\right)^2} + \bar{h}_{\bar{r}\bar{r}} + \frac{1}{\bar{r}}\,\bar{h}_{\bar{r}} \tag{6}$$

This parabolic equation is mathematically identical to the equation for the transient enthalpy distribution as derived by Frind (ref. 8) for an arc column in still air with a step current input where

$$\frac{\partial}{\partial t} \equiv u\,\frac{\partial}{\partial z}$$

Therefore, as z increases, the radial enthalpy distribution approaches the distribution of a steady-state arc column in still air.

For the boundary conditions $h = 0$ at $r = r_e$ and $h = 0$ at $z = 0$, a solution to equation (6) can be obtained by separation of variables since $\bar{h}(\bar{r},\bar{z}) = R(\bar{r})Z(\bar{z})$:

$$R\,\frac{dZ}{d\bar{z}} = \frac{RZ}{\left(\int_0^1 2\pi\bar{r}RZ\,d\bar{r}\right)^2} + Z\,\frac{d^2R}{d\bar{r}^2} + \frac{Z}{\bar{r}}\,\frac{dR}{d\bar{r}}$$

which when divided by RZ is

$$\frac{1}{Z}\,\frac{dZ}{d\bar{z}} - \frac{1}{Z^2\left(\int_0^1 2\pi\bar{r}R\,d\bar{r}\right)^2} = \frac{1}{R}\,\frac{d^2R}{d\bar{r}^2} + \frac{1}{R\bar{r}}\,\frac{dR}{d\bar{r}} = -\beta^2$$

Then

$$\frac{dZ}{d\bar{z}} + \beta^2 Z - \frac{1}{Z\left(\int_0^1 2\pi\bar{r}R\,d\bar{r}\right)^2} = 0 \tag{7}$$

and

$$\frac{d^2R}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{dR}{d\bar{r}} + \beta^2 R = 0 \tag{8}$$

The solution to equation (7) is

$$Z = \left[\frac{1}{\beta^2\left(\int_0^1 2\pi\bar{r}R\ d\bar{r}\right)^2} - \frac{e^{-2\beta^2(\bar{z}+C_1)}}{2\beta^2}\right]$$

From the boundary condition $\bar{h} = 0$ at $\bar{z} = 0$ one obtains

$$Z = \frac{\left(1-e^{-2\beta^2\bar{z}}\right)^{1/2}}{\beta\left(\int_0^1 2\pi\bar{r}R\ d\bar{r}\right)}$$

The solution to equation (8) is

$$R = C_2 J_0(\beta\bar{r}) + C_3 Y_0(\beta\bar{r})$$

From the physical limitation that $\bar{h}$ be finite at $\bar{r} = 0$, C_3 must be zero. From the boundary condition that $\bar{h} = 0$ at $\bar{r} = 1$, β must be the zero of J_0. Since $\bar{h}$ cannot assume negative values, the first zero, 2.4, is chosen. Therefore, a solution to equation (6) is

$$\bar{h} = \frac{1}{2\pi J_1(2.4)}\left[1-e^{-2(2.4)^2\bar{z}}\right]^{1/2} J_0(2.4\bar{r}) \tag{9}$$

From equation (9) it is seen that as the length variable $\bar{z}$ increases without limit,

$$\bar{h} \rightarrow \frac{J_0(2.4\bar{r})}{2\pi J_1(2.4)} = 0.307\ J_0(2.4\bar{r})$$

Therefore, $h_\infty = 0.307\ h_0 J_0(2.4\bar{r})$ which is the approximate solution for the steady state arc in still air obtained by Goldenberg (ref. 7).

Results

The solution derived in the previous section, equation (9), gives the local enthalpy anywhere in the arc column as a function of the radial distance, r, and the axial distance, z. Predictions can now be made of the average enthalpy of the air leaving the column, the local and over-all heat transfer from the column, the local and over-all voltage drops of the column, and the efficiency of the energy exchange processes, while varying the following quantities:

Amperage input

Column radius

Column length

Mass-flow rate

Column pressure

The solution for the local enthalpy in dimensional form is

$$h = 9.43\times10^{-3}\left(\frac{c_p}{kA^{1/2}}\right)\left(\frac{I}{r_e}\right)\left(1-e^{-11.5z/z_o}\right)^{1/2}J_o\left(2.4\frac{r}{r_e}\right) \tag{10}$$

It can be seen from equation (10), first, that enthalpy is directly proportional to the ratio of total current to column radius. Second, it appears that enthalpy increases with decreasing mass flow and/or increasing column length. To explore the trends of efficiency, heat loss, and voltage as the parameters are varied, it is convenient to derive further relationships as follows:

Local heat loss from current-carrying cylinder

$$q = k\left(\frac{\partial T}{\partial r}\right)_{r=r_e} = \frac{k}{c_p}\left(\frac{\partial h}{\partial r}\right)_{r=r_e}$$

where q is the local heat loss per unit area from the column surface and k/c_p is a constant. Upon substitution from equation (10), the above expression becomes

$$q = \frac{k}{c_p}\frac{h_\infty f(z)2.4J_1(2.4)}{r_e}$$

or

$$q = 1.18\times10^{-2}\left(\frac{1}{A^{1/2}}\right)\left(\frac{I}{r_e^2}\right)f(z)$$

where

$$f(z) = \left(1-e^{-11.5z/z_o}\right)^{1/2}$$

Finally,

$$\frac{q}{q_\infty} = f(z)$$

where

$$q_\infty = 1.18\times10^{-2}\left(\frac{1}{A^{1/2}}\right)\frac{I}{r_e{}^2} \tag{11}$$

<u>Local voltage gradient</u>

$$E = \frac{I}{\int_0^{r_e} 2\pi r\sigma \, dr} = \frac{I}{\int_0^{r_e} 2\pi r \, \frac{Ak}{c_p} \, h \, dr} = \frac{2.4}{r_e(AC_c)^{1/2} f(z)}$$

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and

$$\frac{E}{E_\infty} = \frac{1}{f(z)}$$

where

$$E_\infty = \frac{78.0}{r_e A^{1/2}} \tag{12}$$

<u>Radially averaged enthalpy of the air leaving the column</u>

$$h_E = \frac{1}{\pi r_e{}^2}\int_0^{r_e} 2\pi r h \, dr$$

or

$$h_E = 4.08\times10^{-3}\left(\frac{c_p}{kA^{1/2}}\right)\left(\frac{I}{r_e}\right) f(l) \tag{13}$$

where

$$f(l) = \left(1-e^{-11.5l/z_0}\right)^{1/2}$$

<u>Total heat loss from surface of current-carrying cylinder</u>

$$\begin{aligned} Q &= \int_0^l q2\pi r_e \, dz \\ &= \left[2.4\left(\frac{C_c}{A}\right)^{1/2}\frac{I}{r_e}\right]\int_0^l f(z) \\ &= \left[\frac{\dot{w}c_p I}{2.4\pi k r_e}\left(\frac{C_c}{A}\right)^{1/2}\right]\left[\frac{ln\left|\frac{1+f(l)}{1-f(l)}\right|}{2} - f(l)\right] \end{aligned}$$

or

$$\frac{Q}{2\pi r_e q_\infty z_o} = 0.174 \left[\frac{ln \left| \frac{1+f(l)}{1-f(l)} \right|}{2} - f(l) \right] \tag{14}$$

Total voltage drop over arc column

$$V = \int_o^l E \, dz$$

$$= \frac{2.4}{r_e (AC_c)^{1/2}} \int_o^l \frac{1}{f(z)} \, dz$$

$$= \left[\frac{\dot{w} c_p}{2.4\pi k (AC_c)^{1/2} r_e} \right] \left[\frac{ln \left| \frac{1+f(l)}{1-f(l)} \right|}{2} \right]$$

$$\frac{V}{E_\infty z_o} = 0.174 \left[\frac{ln \left| \frac{1+f(l)}{1-f(l)} \right|}{2} \right] \tag{15}$$

One may note that the voltage does not depend upon the current, but only upon the mass flow, $\dot{w}$, the length, l, and the radius, r_e. Voltage would be a decreasing function of current if the curve of σ vs. φ were approximated by $\sigma = A'\varphi^n$ with $n > 1$, and would increase with current if $n < 1$. The choice $n = 1$ results in the independence of voltage on current.

Efficiency with which electric energy is delivered to air leaving column

$$\eta = \frac{\dot{w} h_E}{C_c I^2 \Omega} = \frac{\text{power absorbed by the air}}{\text{power input}}$$

where

$$\Omega = \int_o^l \frac{dz}{\int_o^{r_e} 2\pi r \sigma \, dr}$$

since

$$\sigma = \frac{Ak}{c_p} h$$

then

$$\eta = \frac{\dot{w}\left[\frac{1}{2.4\pi}\left(\frac{C_c}{A}\right)^{1/2}\frac{c_p}{k}\frac{I}{r_e}\right] f(l)}{C_c I^2 \int_0^l \frac{dz}{\int_0^{r_e}\left[2\pi r(AC_c)^{1/2}\frac{I}{r_e}\right]\left[\frac{1}{2\pi J_1(2.4)}\right] f(z) J_o\left(2.4\frac{r}{r_e}\right) dr}}$$

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or

$$\eta = \frac{2f(l)}{ln\left|\frac{1+f(l)}{1-f(l)}\right|} \tag{16}$$

From equation (16) it is noted that η is independent of r_e and is a function of the dimensionless column length, $z/(\dot{w}c_p/\pi k)$, only.

Local properties of the column (eq. (10)) are shown in figures 5 through 7. The dimensionless local enthalpy, $\bar{h}$, is given as functions of the dimensionless axial and radial distances of the column in figures 5 and 6, respectively. These local values can be completely represented by two graphs because the variables $\bar{z}$ and $\bar{r}$ are separable. The radial enthalpy distribution, the Bessel function J_o, is identical to the radial distribution determined by an analogous procedure for a steady state arc in still air. The axial enthalpy distribution approaches an asymptote for large z. Therefore, at large z the radial heat loss becomes equal to the heat input due to ohmic heating.

The dimensionless local heat transfer from the column surface (eq. (11)) is proportional to the center-line enthalpy and varies with dimensionless axial distance identically as enthalpy, as is also seen in figure 5. The dimensionless local voltage gradient (eq. (12)), which is assumed constant on planes perpendicular to the axis, takes very high values at small $\bar{z}$ and rapidly decreases to an asymptotic value (see fig. 7).

The integrated properties of the column are shown in figures 8 and 9. These properties are shown with current held constant. The radially averaged enthalpy of the air leaving the column, h_E (eq. (13)), the total heat loss from the column, Q (eq. (14)), and the total voltage drop through the column, V (eq. (15)), all vary inversely with the radius (fig. 8). The exit air enthalpy increases rapidly with column length where l/z_o is small and approaches an asymptotic value for large l/z_o (fig. 9). The heat loss and voltage drop, however, continually increase with increasing length, and become directly proportional to the length for large l/z_o (fig. 9). The dimensionless column length $l/(\dot{w}c_p/\pi k)$ reflects changes in the dimensional column length and the mass flow rate $\dot{w}$, so that the effects of varying either of these variables can be determined from plots with l/z_o as the abscissa.

Variations in pressure are felt through the changes in the parameters c_p/k and A (see figs. 3 and 4). The term c_p/k does not vary greatly with pressure, so the dimensionless column length l/z_o is not affected greatly by pressure. Also, the term A does not vary greatly with pressure for low enthalpies but does vary appreciably at enthalpies above 17,500 Btu/lb. At high enthalpies the accuracy with which the $\sigma = \sigma(\varphi)$ curve can be approximated by a single straight line is also affected greatly, so precise effects of pressure cannot be determined by this approximate solution. However, qualitatively, an increase in pressure at high enthalpies increases the value of A and decreases the reference values h_∞, q_∞, and E_∞, so that the enthalpy, heat transfer, and voltage should decrease somewhat with increasing pressure, all other parameters remaining unchanged. It should be remarked that pressure changes of several orders of magnitude appear to be necessary to produce significant changes in these reference values, and that the over-all effects of pressure changes on column characteristics are small.

The efficiency (eq. (16)) is a function of the dimensionless column length only, and therefore is independent of the column radius and the current. As shown in figure 10, the efficiency is highest at small l/z_o and continually decreases with increasing l/z_o; thus as the length becomes large, the loss per unit length approaches the power input per unit length. Therefore, the efficiency increases with decreasing column length, l, increasing mass flow rate, $\dot{w}$, or decreasing pressure (increasing c_p/k). One may note that maximum efficiency is not, in general, synonymous with maximum enthalpy. The reason is that at constant finite current the enthalpy tends to zero (eq. (10)) as efficiency tends toward unity with decreasing l/z_o. If the current is allowed to increase without limit, however, while the total power, VI, is held constant, it can be shown from equations (10), (15), and (16) that maximum efficiency occurs in an arc column of vanishingly small length, since the losses then tend toward zero. The solution thus predicts that, for given current and mass-flow rate, the average exit air enthalpy of the arc column can be increased by increasing the column length or decreasing the column radius, that a simultaneous increase in power consumption and heat loss will occur, and that the over-all efficiency will decrease.

Discussion

The analysis presented herein is a treatment of the manner in which air is heated in axial passage through the current-carrying portion of a direct-current electric arc. In establishing a mathematically tractable model for the air-arc interactions, the principal idealizations that radial heat conduction is the only mechanism for heat loss, that the air flow and electric current flow are one-dimensional, and that thermodynamic equilibrium exists at constant pressure have been imposed. Nothing has been said about conditions external to the model except to prescribe at its outer boundary a longitudinal distribution of conductive heat transfer compatible with the cylindrical shape.

It is clear that in practice the current-carrying core of the arc cannot be allowed to come in contact with a solid boundary, whether it be an electrical conductor or an electrical insulator, without disastrous results. Thus, there must exist an annular buffer zone of air between the container wall and the current-carrying core wherein the electrical conductivity approaches zero. The buffer zone will nevertheless possess an appreciable enthalpy and will in fact experience on its inner boundary an enthalpy of about 3,500 Btu/lb (cf. figs. 3 and 4).

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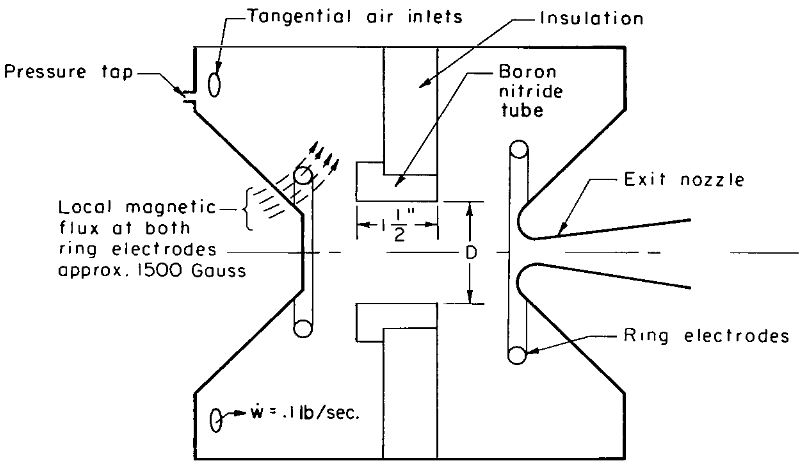
If the outer boundary of the buffer zone, where it is in contact with the container, also is cylindrical, it is clear qualitatively that the temperature of this wall must vary with z in such a manner that the prescribed longitudinal distribution of heat transfer from the core (eq. (11)) can take place. In fact, to preserve the cylindrical core, the enthalpy of the gas at the wall at $z = 0$ must have the core-edge value of 3,500 Btu/lb so that no heat will flow, and must decrease with increasing z to accommodate the radial heat transfer prescribed by equation (11). Since a nonvaporizing physical container cannot exist at the temperature corresponding to the enthalpy at $z = 0$, it follows that, in practice, an arc column in a cold, cylindrical duct at constant temperature must spread with an increase in z.

On the other hand, if the container is maintained at constant temperature, it is easily shown qualitatively that to maintain a cylindrical core the container radius must decrease with z to preserve the required heat-transfer distribution. However, the boundary then cannot be impervious to fluid flow, but must possess a distribution of pores or slots so that the streamlines will be parallel to the axis. It is thus apparent that the boundary conditions at the edge of the current-carrying core which were used in the present analysis are not compatible with boundary conditions which are within practical reach.

It is nevertheless instructive to compare the meager, but sufficiently controlled, experimental information available with some of the integrated results of the present analysis. To this end, the trend of output enthalpy of an electric-arc air heater as a function of the diameter of the arc-column container will be compared with the analytical findings. The justification for so doing, despite the fact that the experimental and theoretical boundary conditions are different, lies in the belief that the detailed differences in local behavior between analytical model and experiment will not lead to gross differences in integrated behavior.

The data, which were collected incidental to a program for development of an electric-arc air heater at the Ames Research Center, consist of a series of runs wherein an arc current on the order of 2,000 amperes and an air flow of 0.1 lb/sec at a pressure of about seven atmospheres were caused to pass through a short, cylindrical tube of boron nitride. The only changes in the arc heater from run to run consisted of changes in tube diameter in the range from 2 to 3-1/2 inches and in the electrode spacing in the range from 3 to 4 inches. The sketch illustrates the experimental arrangement.



A measure of the energy content of the gases leaving the arc chamber through the choked nozzle approximately 2 inches downstream of each boron nitride tube was obtained from a Mollier chart for equilibrium air. Inputs for the enthalpy computation were the measured mass flow, the measured arc chamber pressure, the measured diameter of the sonic discharge throat, and the nozzle coefficient measured in cold flow. To minimize effects of contamination due to ablation of boron nitride, the runs were of approximately 1-second duration. These data are given in the list below.

Hole diameter, D, ft	0.292	0.250	0.229	0.167
Current, I, amp	1400	2050	2450	2150
Energy content, H, Btu/lb	4700	6000	6200	7200

To allow a direct comparison of these experimental data with the analytical results, the average energy content of the air between the arc column axis and the boron nitride tube wall at the tube exit must be calculated. To approximate the energy content of the air which bypassed the arc column the assumption ρu = constant was extended to the tube wall and a constant value of radial heat flux was assumed between the arc column and the wall at the tube exit. The approximate enthalpy profile at the exit is, therefore,

$$h = h_\infty J_o\left(2.4\,\frac{r}{r_e}\right) f(l) \qquad \text{for} \quad 0 \leq r \leq r_e \tag{17}$$

and

$$h = -1.25 h_\infty f(l)\, ln\left(\frac{r}{r_e}\right) \qquad \text{for} \quad r \geq r_e \tag{18}$$

The average enthalpy, H, to be compared with the experimental data is therefore

$$H = \frac{4}{\pi D^2}\int_o^{D/2} 2\pi r \tilde{h}\, dr$$

or

$$H = 3500 + h_\infty f(l)\left\{0.433\left(\frac{2r_e}{D}\right)^2 - 1.25\left[ln\left(\frac{D}{2r_e}\right) - \frac{1}{2} + 2\left(\frac{r_e}{D}\right)^2\right]\right\} \tag{19}$$

where

$$h_\infty f(l) = 9.43\times10^{-3}\left(\frac{c_p}{kA^{1/2}}\right)\left(\frac{I}{r_e}\right)\left[1-e^{-11.5\,l/z_o}\right]^{1/2}$$

The tube radius, D/2, to be compared with experiment is

$$\frac{D}{2} = r_e \exp\left[-\,0.80\,\frac{h_w}{h_\infty f(l)}\right] \tag{20}$$

Values of the various parameters used in this calculation and their sources were as follows:

$l = 0.125$ ft (experiment - see sketch)

$\dot{w} = 0.1$ lb/sec (experiment - see sketch)

$c_p/k = 10^4$ ft sec/lb (fig. 3)

$A = 200$ mho-sec/Btu (fig. 4)

$C_c = 0.947\times10^{-3}$ Btu/watt-sec by definition

The value of the gas enthalpy at the tube wall is unknown; so two values, $\tilde{h} = 0$ and $\tilde{h} = 600$ Btu/lb, are used as limiting values. The comparison of the experimental data with the curves obtained with the values above is shown in figure 11.

It can be observed that the data are in gratifyingly good agreement with the calculation which assumes that the boron nitride tubes were near their sublimation temperatures. Whether or not this agreement is merely fortuitous cannot now be decided, because data giving voltage drops across the length of the tube and heat flows to the wall are lacking. The results do serve to show, however, that significant gains in output enthalpy of a current-limited plasma generator can be realized by increase of the ratio of current per unit circumference of the arc column.

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The analysis further indicates that increases of output enthalpy will accrue if the ratio of column length to mass flow is made as large as is feasible. Since it is in general not desirable to decrease mass flow, one is driven to consider longer arc columns if high enthalpy is to be realized.

The extent to which the mass flow rate and the length can be increased before the onset of turbulence or before an unstable mode of operation sets in cannot be predicted by this theory, but it appears likely that a simultaneous reduction of diameter, increase of length, and increase of mass flow would tend to induce a turbulent mode of operation or would drive the column off the axis of symmetry or both. Therefore, attempts to simply "scale up" generators by following the predictions of this analysis should be made with caution.

CONCLUSIONS

The enthalpy and electric conductivity distributions of an arc column in air moving along the axis can be approximated with an analytical solution if the flow can be considered laminar and steady with heat conduction the dominant form of energy transfer. This solution predicts the changes in the arc column enthalpy, efficiency, heat transfer, and voltage when the column length, radius, mass-flow rate, current, and pressure are varied. The enthalpy for an arc column with a given flow rate and current supply can be increased by decreasing the radius of the arc column and increasing the length, and a concurrent increase in voltage and radial heat loss due to conduction will be experienced. The trend of increasing enthalpy output as radius is decreased agrees with experiment.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., May 24, 1962

APPENDIX

LINEARIZING APPROXIMATIONS OF THE ENERGY EQUATION

Goldenberg (ref. 7) obtained an analytic approximate solution to the Elenbaas-Heller energy equation by linearizing the relation between the dependent variables σ and φ. The Elenbaas-Heller energy equation is

$$\sigma E^2 = -\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right)$$

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or

$$\sigma E^2 = -\frac{1}{r}\varphi_r - \varphi_{rr}$$

The piecewise linear approximations are $\sigma = A_n\varphi + B_n$ for the n approximating segments. These linear approximations of the $\sigma - \varphi$ curve are shown in figure 12. The Elenbaas-Heller energy equation then reduces to

$$\varphi_{rr} + \frac{1}{r}\varphi_r + E^2(A\varphi + B) = 0$$

for which the solutions are

$$\varphi = -(B/A) + C_1J_o(rE\sqrt{A}) + C_2Y_o(rE\sqrt{A}) \qquad \text{for} \quad A > 0$$

$$\varphi = -(Br^2E^2/4) + C_1 \ln\,(rE + C_2) \qquad \text{for} \quad A = 0$$

$$\varphi = -(B/A) + C_1I_o(rE\sqrt{-A}) + C_2K_o(rE\sqrt{-A}) \qquad \text{for} \quad A < 0$$

$J_o(x)$ = Bessel function of first kind

$Y_o(x)$ = Bessel function of second kind

$I_o(x)$ = Modified Bessel function of first kind

$K_o(x)$ = Modified Bessel function of second kind

The segments of φ are joined so that φ is continuous and the heat transfer (φ_r) is continuous. The analytical solutions with the approximations of figure 12 are compared with King's numerical solution (7) in figure 13. Even rough approximations of the $\sigma - \varphi$ curve give reasonably close approximations to the temperature profile as shown by the approximations II and III (figs. 12 and 13). With such rough approximations, local irregularities, such as the humps in the temperature profile discussed by King (ref. 9), are smoothed out and not predicted. However, the over-all trends of the solution are not masked.

For the energy equation of the model of the arc column in moving air (eq. (3)), the approximation that $\sigma = A\varphi$ (which is the Goldenberg approximation with $\varphi = 0$ at $\sigma = 0$) and $h = (c_p/k)\ \varphi$ (where c_p/k is constant) transforms the equation to equation (5) which can also be written as

$$\bar{\varphi}_{\bar{z}} = \frac{\bar{\varphi}}{\left(\int_0^1 2\pi\bar{r}\bar{\varphi}\,d\bar{r}\right)^2} + \left[\frac{r_e}{z_o}\right]^2 \bar{\varphi}_{\bar{z}\bar{z}} + \bar{\varphi}_{\bar{r}\bar{r}} + \frac{1}{\bar{r}}\,\bar{\varphi}$$

The solution for this with $z_o \gg r_e$ is equation (9):

$$\bar{\varphi} = \bar{h} = f(\bar{z})\ g(\bar{r})$$

The function $g(r)$ is precisely the Goldenberg approximate solution which already was shown to give a reasonable approximation of the radial φ profile. The axial profile is a direct function of the energy absorbed per unit length, and the errors can be judged from the following form of the energy equation.

Rate of energy absorbed = Ohmic heating - Energy lost

$$= \frac{I^2}{\int_0^R 2\pi r\sigma\,dr} - 2\pi r\varphi_r$$

The σ and φ radial profiles are reasonably approximated by the Goldenberg solution, so both of the above terms are also reasonably approximated. Therefore, the axial profile (rate of energy absorbed) retains the general trends of the numerical solutions and can be used to predict gross behavior of the arc column.

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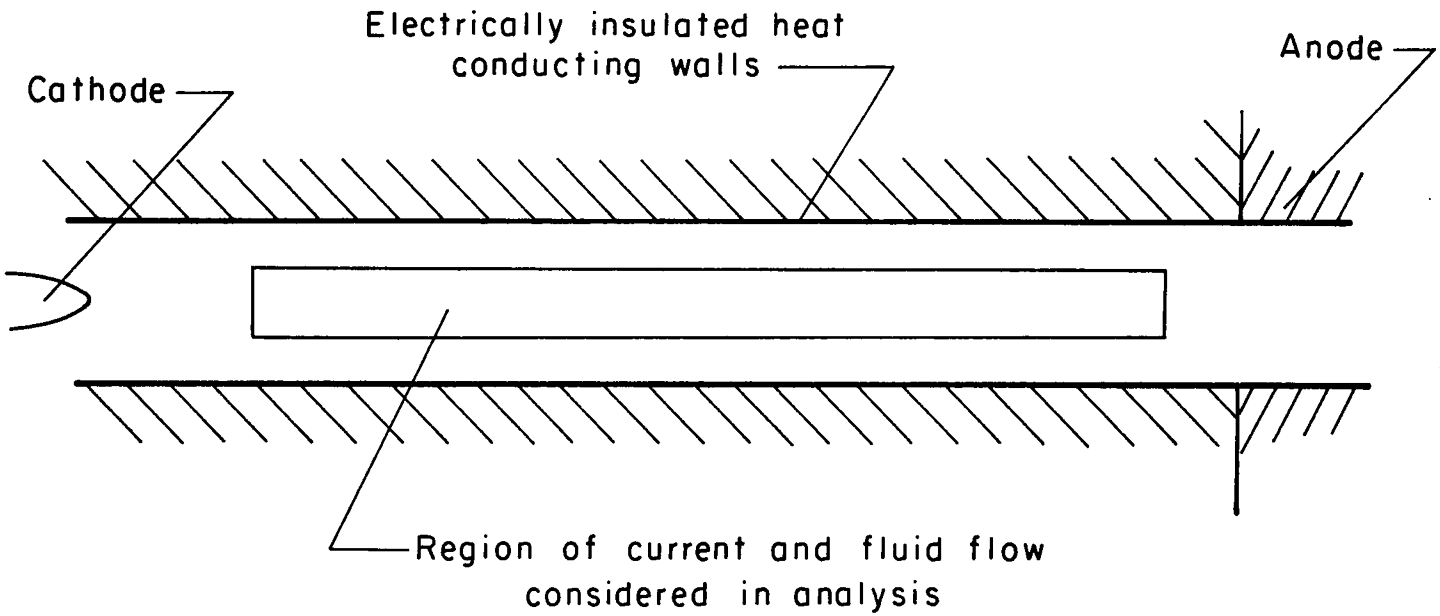

Figure 1.- Flow region of arc-jet plasma generator considered in analysis.

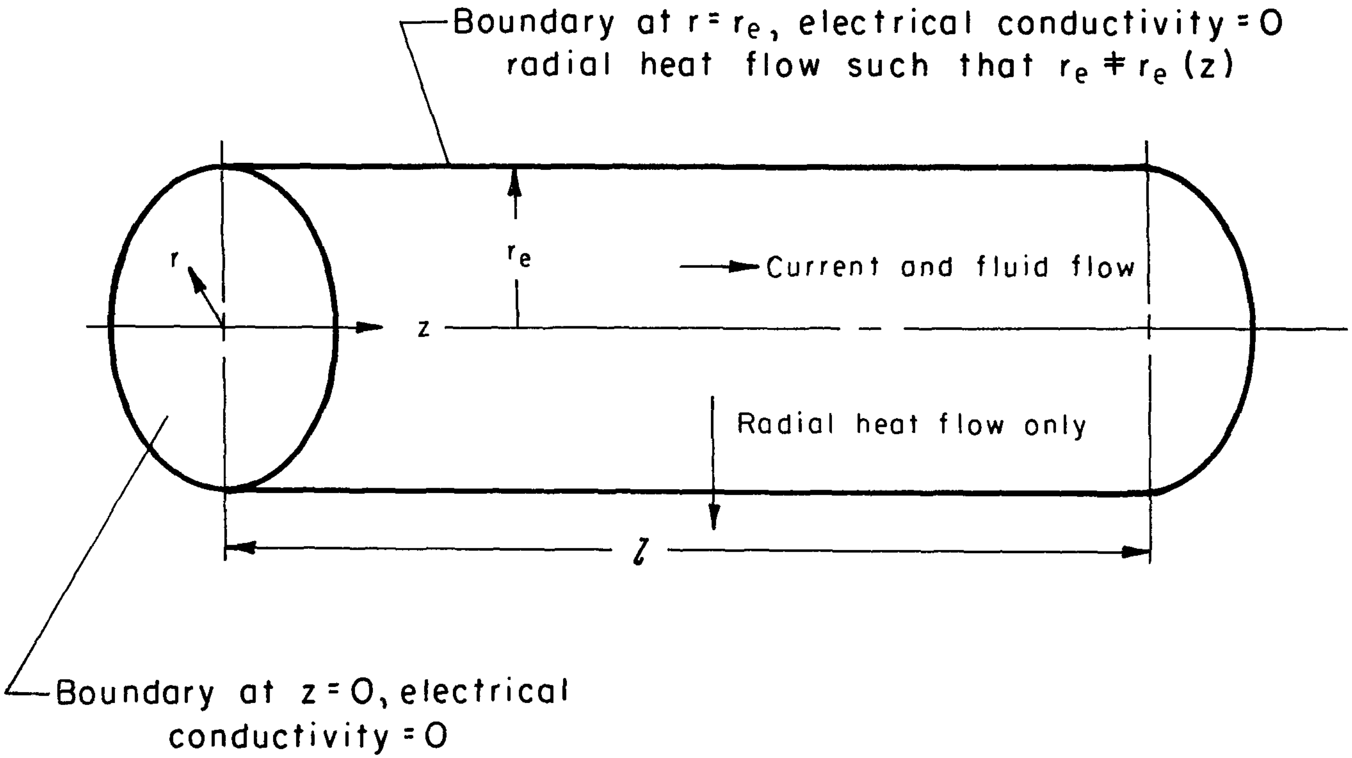

Figure 2.- Model of the arc column.

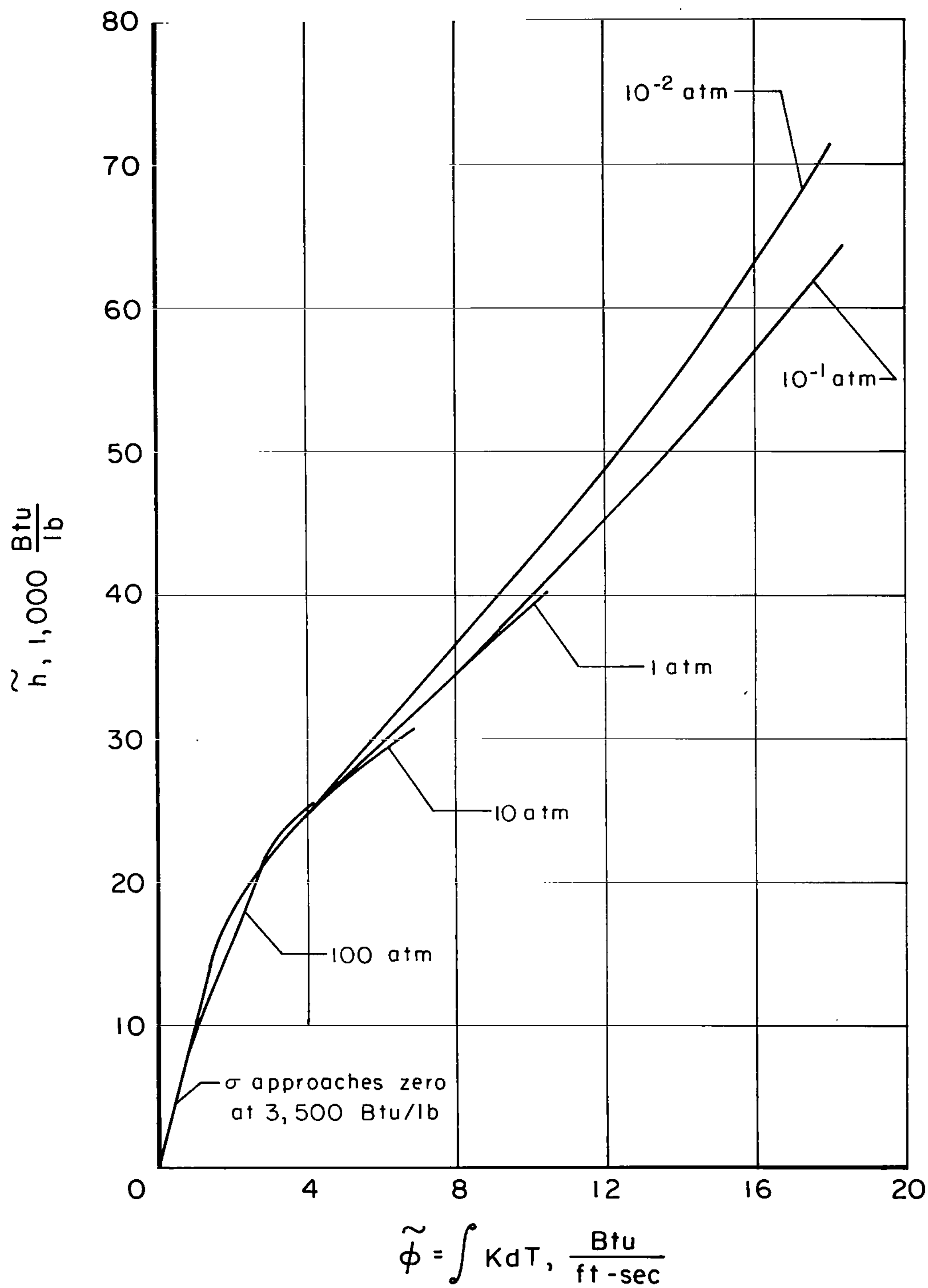

Figure 3.- Relationship between enthalpy and conduction function for air in thermodynamic equilibrium (ref. 5).

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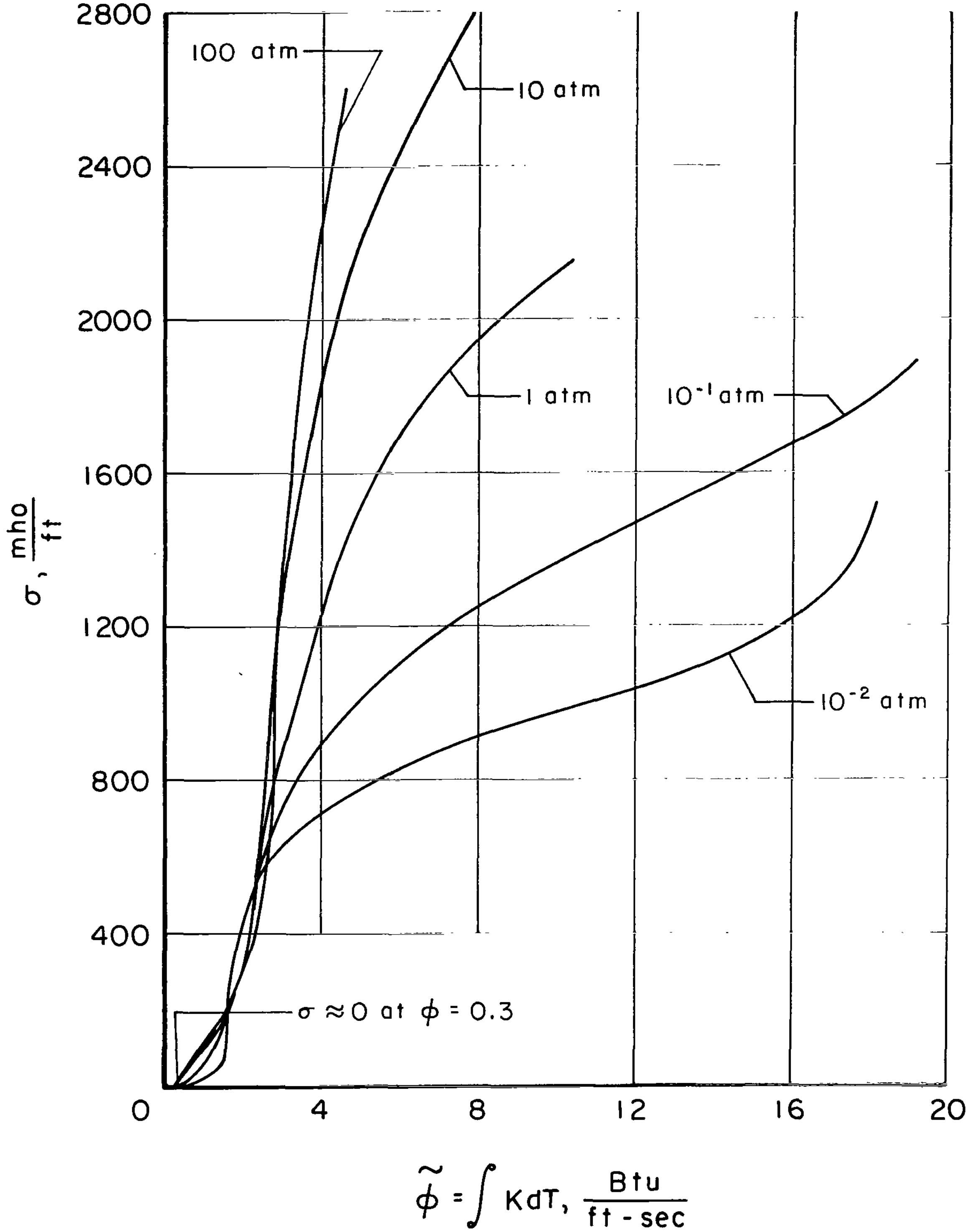

Figure 4.- Relationship between electric conductivity and conduction function for air in thermodynamic equilibrium (refs. 5 and 6).

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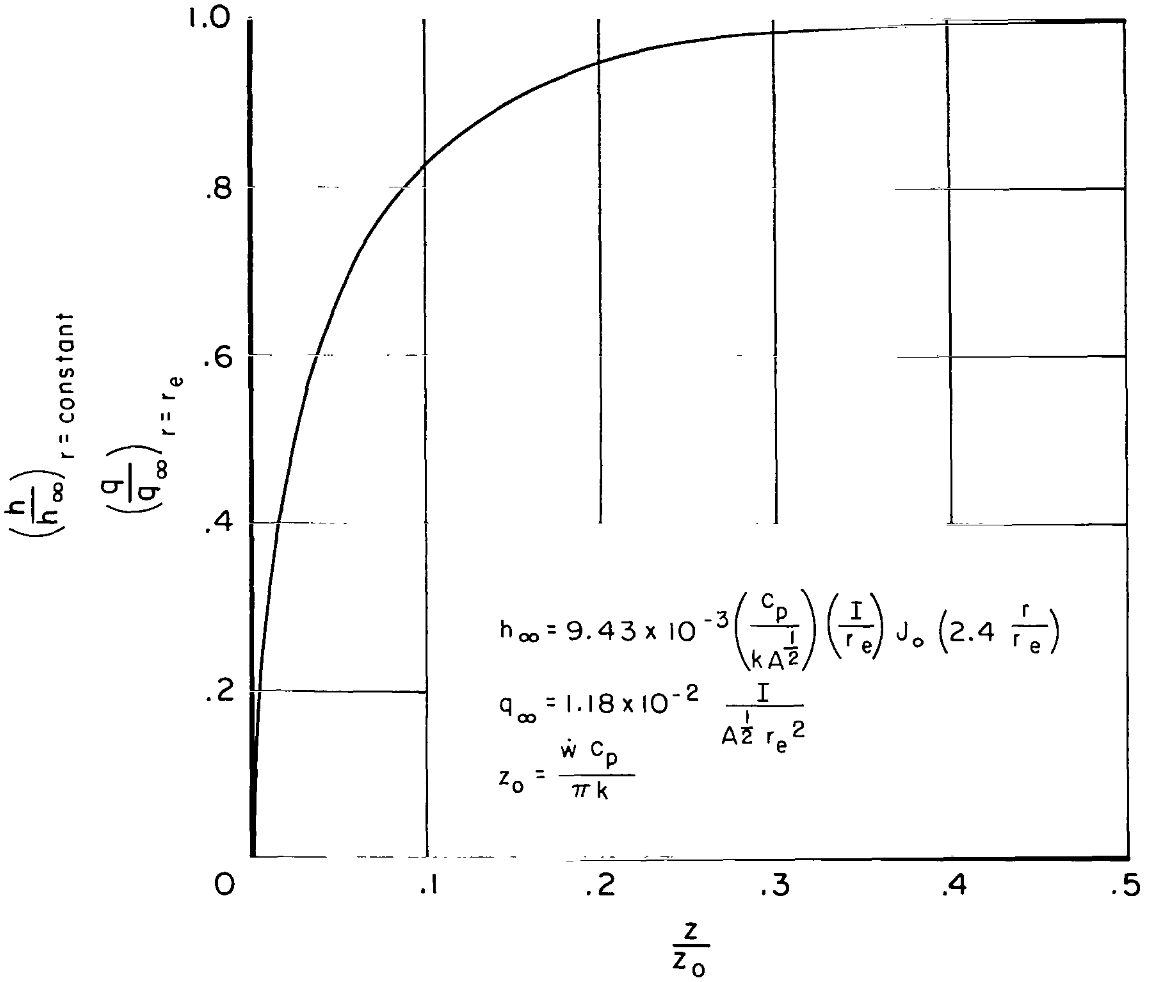

Figure 5.- Axial distribution of enthalpy and longitudinal distribution of local radial heat transfer.

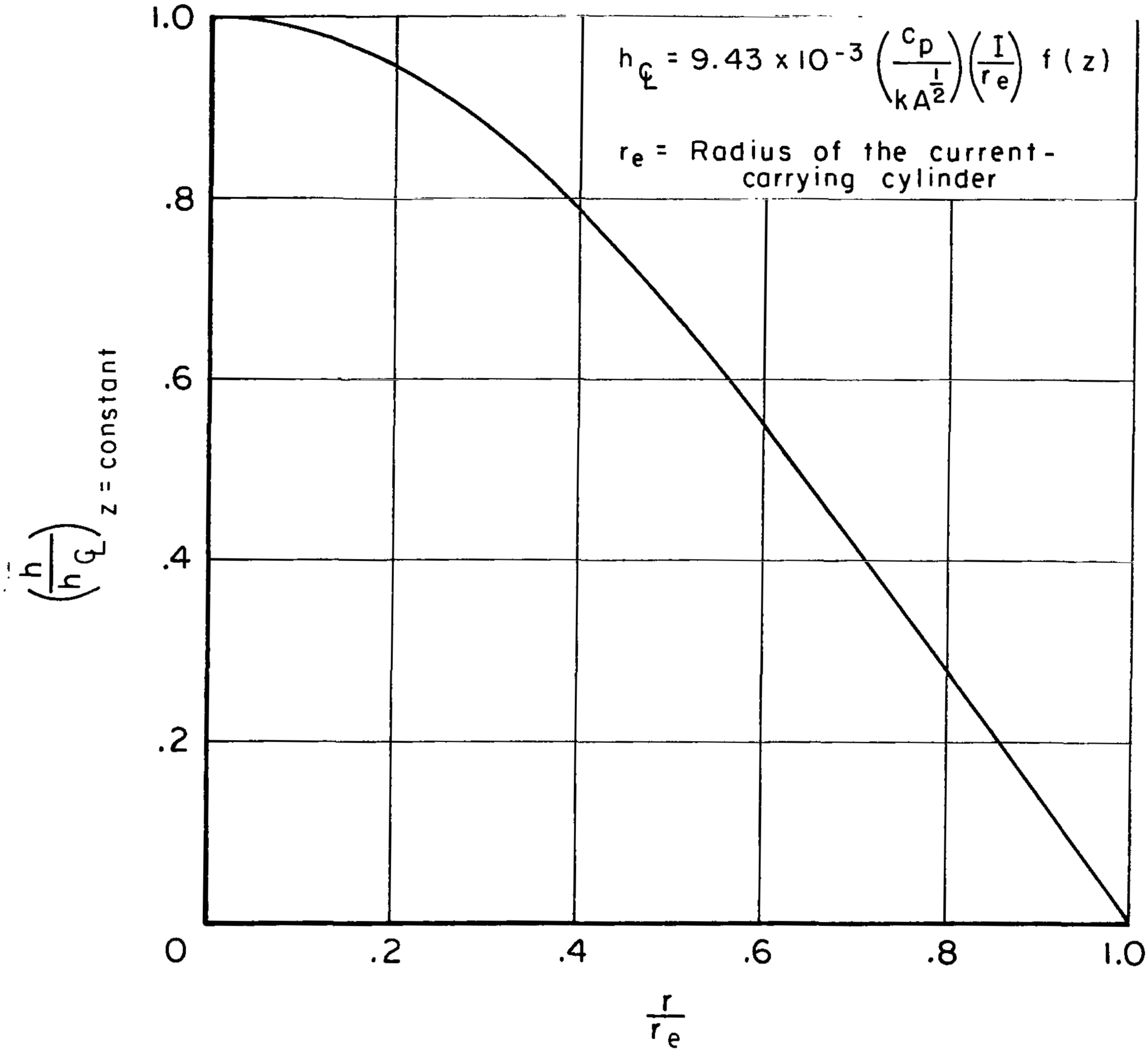

Figure 6.- Radial distribution of enthalpy.

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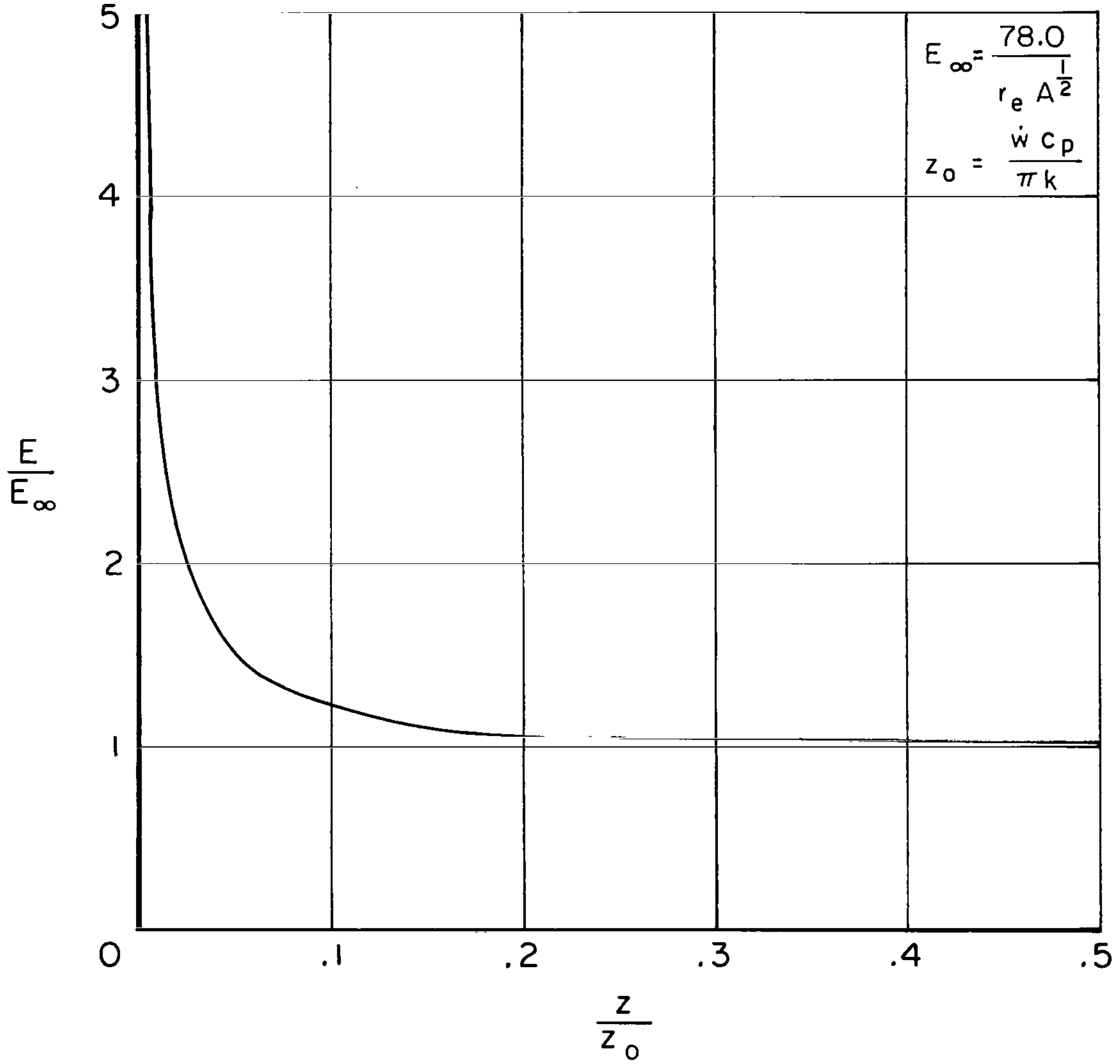

Figure 7.- Longitudinal distribution of local voltage gradient.

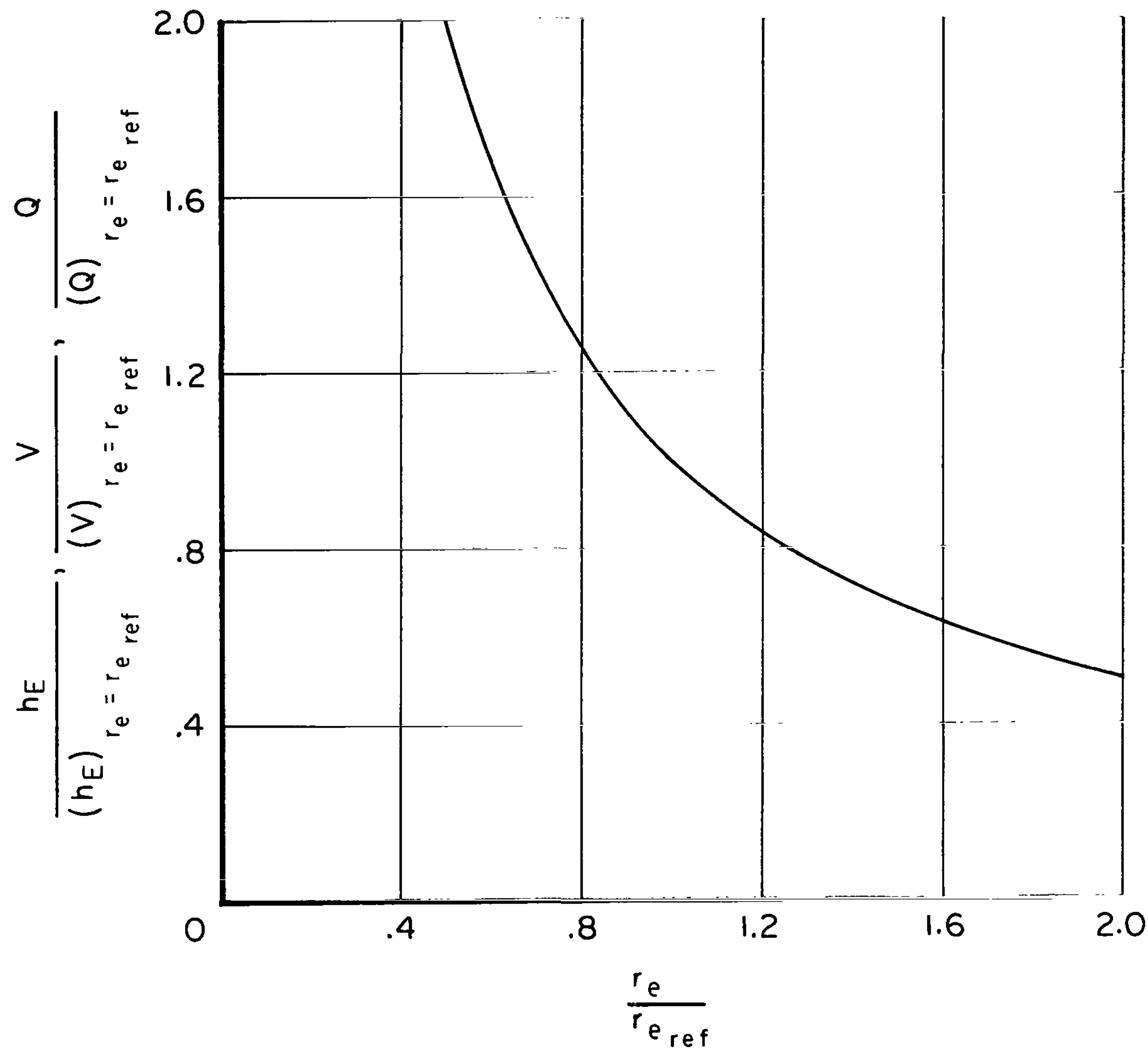

Figure 8.- Variation of integrated column properties with column radius (constant current).

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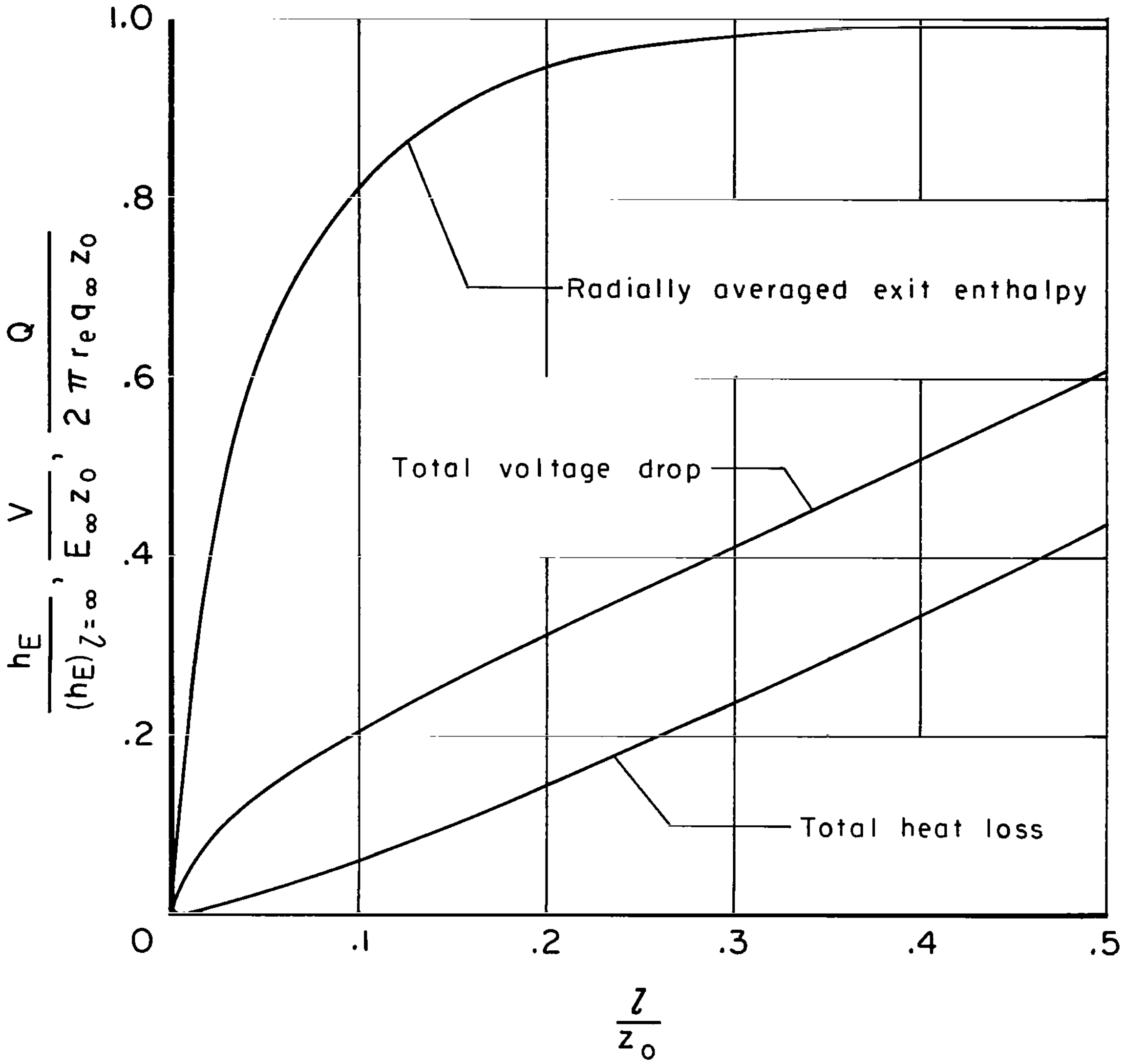

Figure 9.- Variation of integrated column properties with dimensionless column length (constant current).

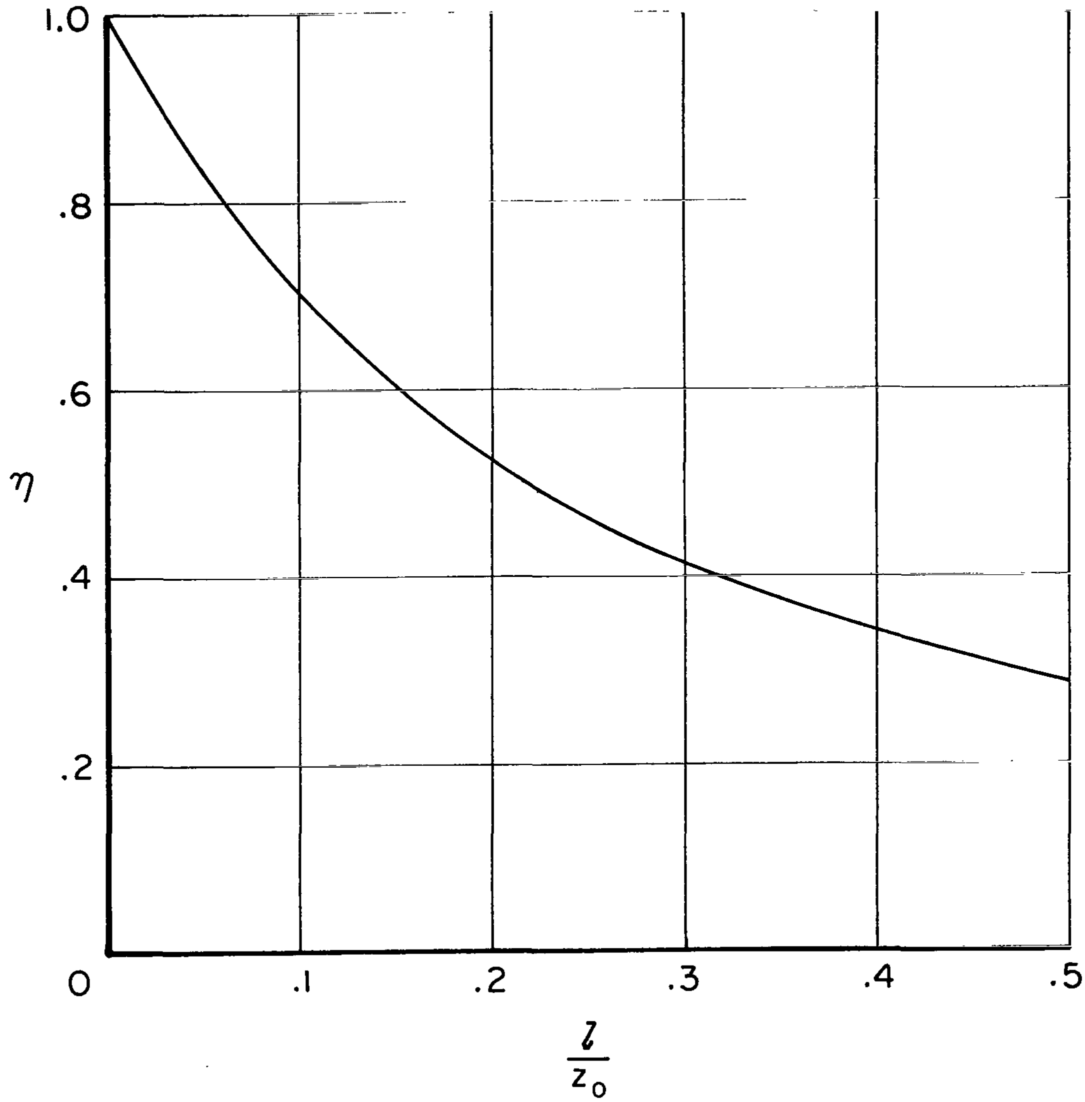

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Figure 10.- Variation of arc column efficiency with dimensionless column length.

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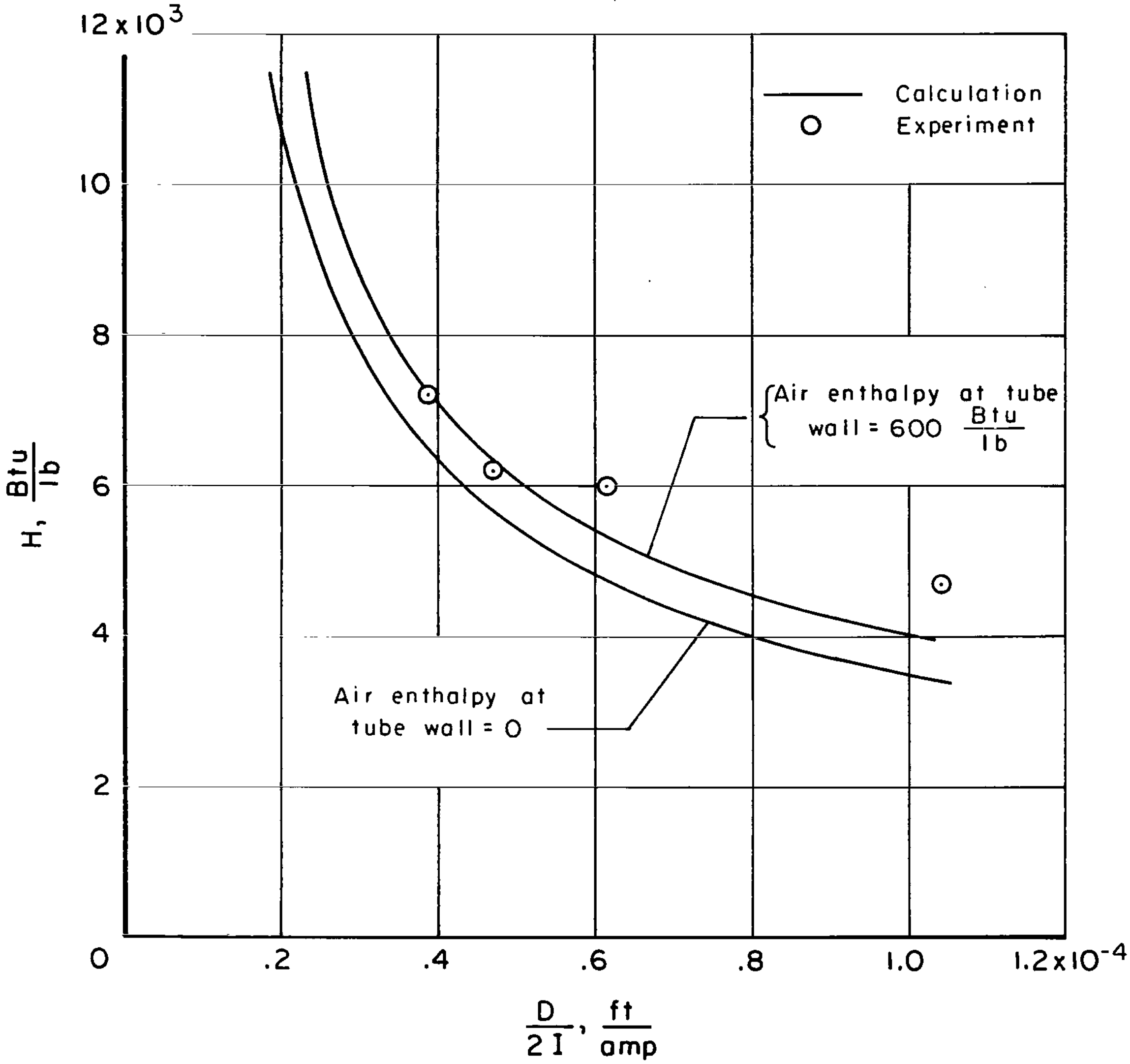

Figure 11.- Effect of variation of column diameter and current on average output enthalpy.

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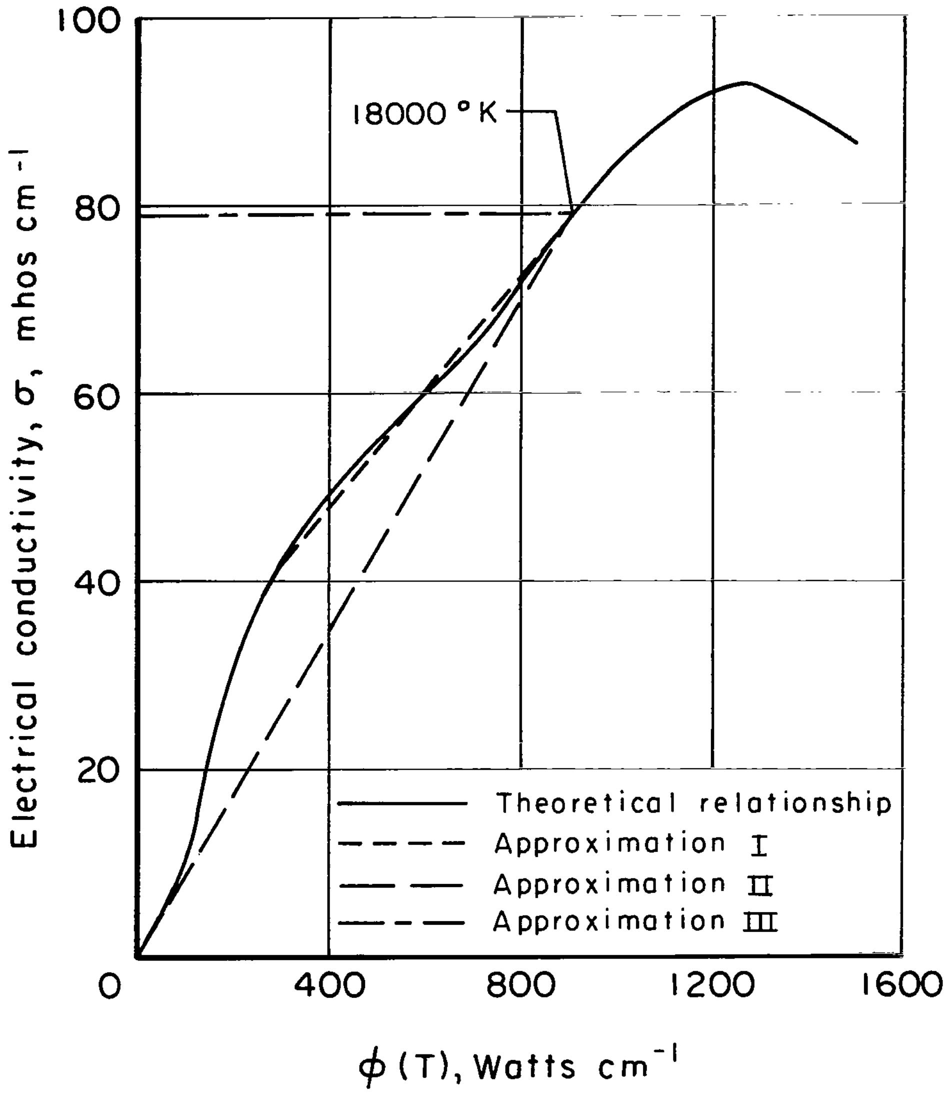

Figure 12.- Relationship between dependent variables of the Elenbaas energy equation for nitrogen at 1 atmosphere (from ref. 7).

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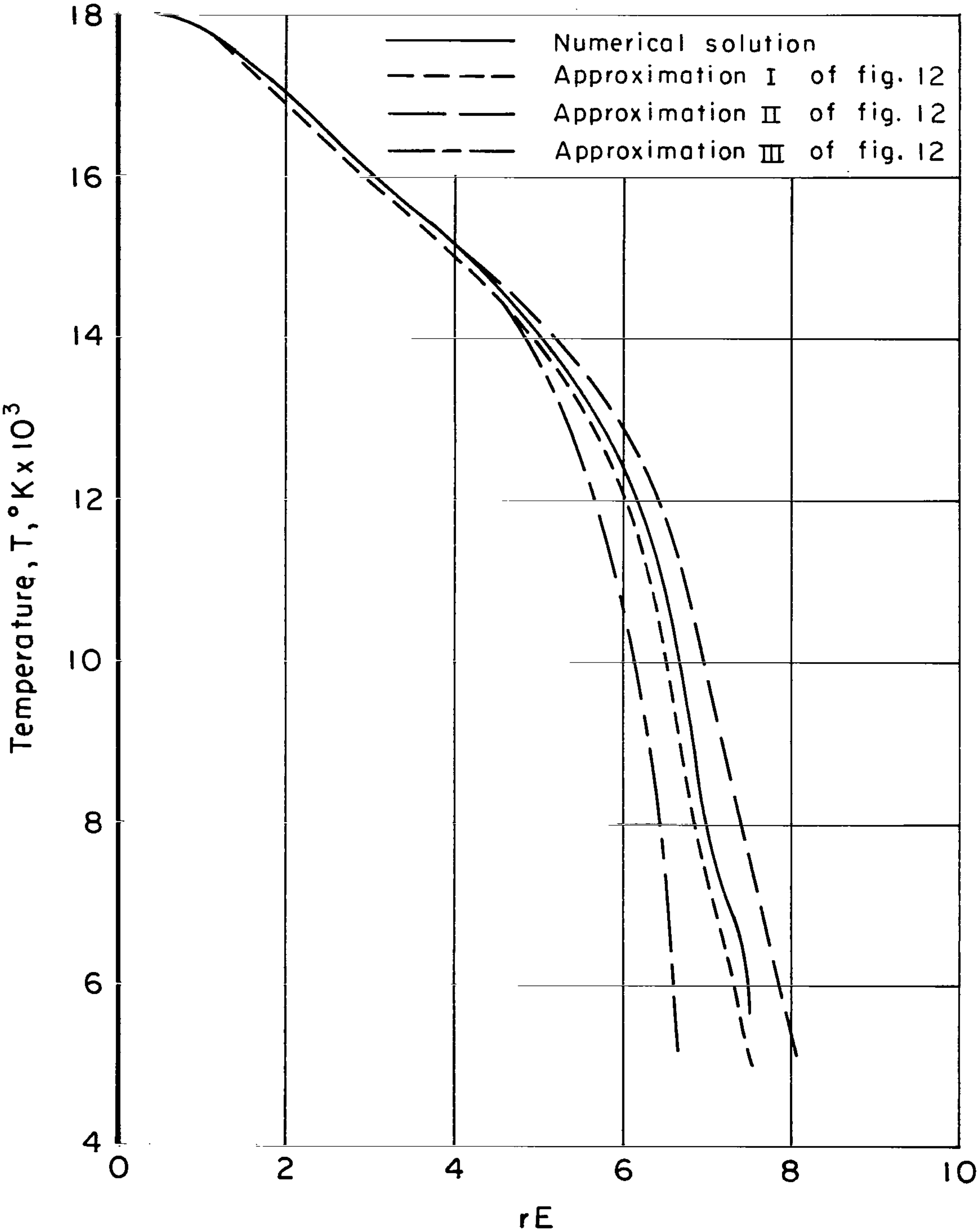

Figure 13.- Comparison of analytic approximate solutions and the numerical solution of the Elenbaas energy equation (from ref. 7).

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